

The forward-backward algorithm and the normal problem

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Abstract

The forward-backward splitting technique is a popular method for solving monotone inclusions that has applications in optimization. In this paper we explore the behaviour of the algorithm when the inclusion problem has no solution. We present a new formula to define the normal solutions using the forward-backward operator. We also provide a formula for the range of the displacement map of the forward-backward operator. Several examples illustrate our theory.

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1 Introduction

Throughout this paper we work under the assumption that

X is a real Hilbert space,

with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. A (possibly) set-valued operator $A : X \rightrightarrows X$ is *monotone* if any two pairs (x, u) and (y, v) in the graph of A satisfy $\langle x - y, u - v \rangle \geq 0$, and is *maximally monotone* if it is monotone and any proper enlargement of the graph of A (in terms of set inclusion) will no longer preserve the monotonicity of A . In the following we assume that

$A : X \rightrightarrows X$ and $B : X \rightrightarrows X$ are maximally monotone operators. (1)

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Thanks to the fact that the *subdifferential* operator associated with a convex lower semicontinuous proper function is a maximally monotone operator (see Fact 3.6 below), the notion of monotone operators becomes of significant importance in optimization and nonlinear analysis. For further discussion on monotone operator theory and its connection to optimization see, e.g., the books [8], [17], [19], [21], [44], [45], [49], [50], and [51].

The problem of finding a zero of the sum of two maximally monotone operators A and B is to find $x \in X$ such that $x \in (A + B)^{-1}0$. When specializing A and B to subdifferential operators of convex lower semicontinuous proper functions, the problem is equivalent to finding a minimizer of the sum of the two functions, which is a classical optimization problem.

Suppose that A is *firmly nonexpansive*¹ (see Section 2). Let $x_0 \in X$ and let T_{FB} be the forward-backward operator associated with the pair (A, B) (see Section 3). When $(A + B)^{-1}0 \neq \emptyset$ the sequence $(T_{\text{FB}}^n x_0)_{n \in \mathbb{N}}$ produced by iterating the forward-backward operator converges weakly² to a point in $(A + B)^{-1}0 = \text{Fix } T_{\text{FB}} = \{x \in X \mid x = T_{\text{FB}}x\}$ (see, e.g., [47], [33] or [23]). Applications of this setting appear in convex optimization (see, e.g., [8, Section 27.3]), evolution inclusions (see, e.g., [2]) and inverse problems (see, e.g., [24] and [25]).

The goal of this work is to examine the forward-backward operator in the inconsistent case, i.e., when $(A + B)^{-1}0 = \emptyset$, using the framework of the normal problem introduced in [12]. In this case $\text{Fix } T_{\text{FB}} = \emptyset$, and the classical analysis, which uses the advantage of iterating an averaged operator (see Section 2 below) that has a fixed point, is no longer applicable.

Let us summarize the main contributions of the paper:

- R1** We provide a systematic study of the forward-backward operator when the sum problem is possibly inconsistent. This is mainly illustrated in Proposition 4.1 where we establish the connection between the perturbed problem introduced in [12] and the forward-backward operator.
- R2** We prove that the range of the displacement operator associated with the forward-backward operator T_{FB} coincides with that of the Douglas-Rachford operator T_{DR} . Consequently, the minimal displacement vectors associated with T_{FB} and T_{DR} coincide (see Theorem 4.2). This gives an alternative approach to define the normal problem introduced in [12].
- R3** A significant consequence of **R2** is that it allows to use the advantage of the self-duality of T_{DR} (which does not hold for T_{FB} as we illustrate in Example 4.11) to draw more conclusions about T_{FB} . In particular, in Theorem 5.3 we provide a formula for the range of the displacement operator in terms of the ranges of the underlying operators using the notion of *near equality*. The result simplifies to more elegant formulae when specializing the operators to subdifferential operators as illustrated in Proposition 5.7. Our results are sharp in the sense that near equality cannot be replaced by equality which we illustrate in Example 5.4.

¹We point out that the assumption of that A is firmly nonexpansive can be relaxed to A is cocoercive (see Remark 3.1).

²For general conditions on *strong* convergence of the forward-backward algorithm we refer the reader to [2].

R4 In the case when A and B are affine, we prove that, in the consistent case, the sequence produced by iterating T_{FB} converges *strongly* to the *nearest* point in the set of zeros of the sum. If X is finite-dimensional, we also get *linear* rate of convergence (see Theorem 6.6).

The remainder of this paper is organized as follows: Section 2 provides facts and auxiliary results concerning averaged and (firmly) nonexpansive operators. In Section 3, we provide an overview of the Attouch-Théra duality and formulate the primal and dual solutions using the forward-backward operator. Our main results start in Section 4, which deals with the normal problem and the connection to the forward-backward operator. In Section 5, we explore the range of the displacement operator associated with the forward-backward operator. In Section 6, we study the asymptotic behaviour of *asymptotically regular* affine nonexpansive operators in the possibly fixed point free setting. An application to the forward-backward algorithm is provided as well. Finally in Section 7 we provide some algorithmic consequences.

Notation

Let C be a nonempty closed convex subset of X . We use ι_C , N_C and P_C to denote the *indicator function*, the *normal cone* operator and the *projector* (this is also known as *nearest point mapping*) associated with C , respectively. Let $f : X \rightarrow]-\infty, +\infty]$ be convex, lower semicontinuous, and proper. The *subdifferential* of f is the (possibly) set-valued operator $\partial f : X \rightrightarrows X : x \rightarrow \{u \in X \mid (\forall y \in X) f(y) \geq f(x) + \langle u, y - x \rangle\}$. Let $\text{Id} : X \rightarrow X$ be the identity operator. The *resolvent* of A is $J_A := (\text{Id} + A)^{-1}$ and the *reflected resolvent* is $R_A := 2J_A - \text{Id}$. Otherwise, the notation we adopt is standard and follows, e.g., [8] and [40].

2 Averaged and (firmly) nonexpansive operators

Let $T : X \rightarrow X$. Then T is *nonexpansive* if

$$(\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\| \leq \|x - y\|; \quad (2)$$

T is *firmly nonexpansive* if

$$(\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\|^2 + \|(\text{Id} - T)x - (\text{Id} - T)y\|^2 \leq \|x - y\|^2; \quad (3)$$

and T is *averaged* if there exists $\alpha \in]0, 1[$ and a nonexpansive operator $N : X \rightarrow X$ such that

$$T = (1 - \alpha) \text{Id} + \alpha N. \quad (4)$$

Fact 2.1. *The following hold:*

- (i) J_A is single-valued, maximally monotone and firmly nonexpansive.
- (ii) (The inverse resolvent identity) $J_{A^{-1}} = \text{Id} - J_A$.

Proof. (i): See [35, Corollary on page 344] and [41, Proposition 1(c)]. (ii): See, e.g., [40, Lemma 12.14]. ■

In the sequel we make use of the useful characterization (see, e.g., [31, Equation 11.1 on page 42]):

$$T \text{ is firmly nonexpansive} \Leftrightarrow (\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle. \quad (5)$$

Definition 2.2 (asymptotic regularity of operators vs. sequences). Let $T : X \rightarrow X$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X . Then T is asymptotically regular if $(\forall x \in X) \ T^n x - T^{n+1} x \rightarrow 0$ and $(x_n)_{n \in \mathbb{N}}$ is asymptotically regular if $x_n - x_{n+1} \rightarrow 0$.

Fact 2.3. Suppose that $T : X \rightarrow X$ is averaged; in particular, firmly nonexpansive. Then T is asymptotically regular.

Proof. See [20, Corollary 1.1 & Proposition 2.1] or [8, Proposition 5.15(ii) & Corollary 5.16(ii)]. ■

Fact 2.4. Suppose that $T : X \rightarrow X$ is nonexpansive. Then $\overline{\text{ran}}(\text{Id} - T)$ is nonempty closed and convex. Consequently the minimal displacement vector associated with T is the unique well-defined vector

$$v_T := P_{\overline{\text{ran}}(\text{Id} - T)} 0. \quad (6)$$

Proof. See [3], [20] or [38]. ■

Unless otherwise stated, throughout this paper we assume that

$$T : X \rightarrow X \text{ is nonexpansive.}$$

The following result is well-known when T is firmly nonexpansive. We include a simple proof, when T is averaged, for the sake of completeness (see also [10, Lemma 3.9]).

Proposition 2.5. Suppose that T is averaged and that $v_T := P_{\overline{\text{ran}}(\text{Id} - T)} 0 \in \text{ran}(\text{Id} - T)$. Let $x \in X$. Then the following hold:

- (i) $\sum_{n=0}^{\infty} \|T^n x - T^{n+1} x - v_T\|^2 < +\infty$.
- (ii) $T^n x - T^{n+1} x \rightarrow v_T$, equivalently; the sequence $(T^n x + n v_T)_{n \in \mathbb{N}}$ is asymptotically regular.

Proof. It follows from [23, Lemma 2.1] that $(\exists \alpha \in]0, 1[)$ such that $(\forall x \in X) (\forall y \in X)$

$$\|(\text{Id} - T)x - (\text{Id} - T)y\|^2 \leq \frac{\alpha}{1 - \alpha} (\|x - y\|^2 - \|Tx - Ty\|^2). \quad (7)$$

Moreover [6, Proposition 2.5(vi)] implies that $(T^n x + n v_T)_{n \in \mathbb{N}}$ is Fejér monotone with respect to $\text{Fix}(v_T + T)$. Now let $n \in \mathbb{N}$ and let $y_0 \in \text{Fix}(v_T + T)$. Using [6, Proposition 2.5(iv)] we learn that $T^n y_0 = y_0 - n v_T$. It follows from (7) applied with (x, y) replaced by $(T^n x, T^n y_0)$ that

$$\|T^n x - T^{n+1} x - v_T\|^2 = \|(\text{Id} - T)T^n x - (\text{Id} - T)T^n y_0\|^2 \quad (8a)$$

$$\leq \frac{\alpha}{1 - \alpha} (\|T^n x - T^n y_0\|^2 - \|T^{n+1} x - T^{n+1} y_0\|^2). \quad (8b)$$

(i): This follows from (8) by telescoping. (ii): This is a direct consequence of (i). ■

Proposition 2.6. Suppose that $v_T := P_{\overline{\text{ran}}(\text{Id}-T)}0 \in \text{ran}(\text{Id}-T)$ and that $\text{int Fix}(v_T + T) \neq \emptyset$. Then the following hold:

- (i) $\sum_{n=0}^{\infty} \|T^n x - T^{n+1} x - v_T\| < +\infty$.
- (ii) $(T^n x + n v_T)_{n \in \mathbb{N}}$ converges strongly.

Proof. The proof follows along the lines of [8, Proposition 5.10]. (i): Let $x \in \text{Fix}(v_T + T)$ and let $r > 0$ such that $\text{ball}(x; r) \subseteq \text{Fix}(v_T + T)$. Obtain a sequence $(y_n)_{n \in \mathbb{N}}$ defined as:

$$(\forall n \in \mathbb{N}) \quad y_n = \begin{cases} x, & \text{if } x_{n+1} = x_n; \\ x - r \frac{x_{n+1} - x_n}{\|x_{n+1} - x_n\|}, & \text{otherwise.} \end{cases} \quad (9)$$

Then $(y_n)_{n \in \mathbb{N}} \subseteq \text{ball}(x; r)$. Set $(\forall n \in \mathbb{N}) \quad x_n := T^n x + n v_T$. It follows from [6, Proposition 2.5(vi)] that the sequence $(x_n)_{n \in \mathbb{N}}$ is Fejér monotone with respect to $\text{Fix}(v + T)$, therefore $(\forall n \in \mathbb{N}) \quad \|x_{n+1} - y_n\|^2 \leq \|x_n - y_n\|^2$; equivalently $(\forall n \in \mathbb{N}) \quad \|x_{n+1} - x + (x - y_n)\|^2 \leq \|x_n - x + (x - y_n)\|^2$. Expanding and simplifying in view of (9) yield $(\forall n \in \mathbb{N}) \quad \|x_{n+1} - x\|^2 \leq \|x_n - x\|^2 - 2\langle x_n - x_{n+1}, x - y_n \rangle = \|x_n - x\|^2 - 2r\|x_n - x_{n+1}\|$. Telescoping yields

$$\sum_{n=0}^{\infty} \|x_n - x_{n+1}\| \leq \frac{1}{2r} \|x_0 - x\|^2. \quad (10)$$

(ii): It follows from (10) that $(x_n)_{n \in \mathbb{N}} = (T^n x + n v_T)_{n \in \mathbb{N}}$ is a Cauchy sequence and therefore it converges. ■

Let S be nonempty subset of X and let $a \in X$. Before we proceed further we need the following useful translation formula (see, e.g., [8, Proposition 3.17]).

$$(\forall x \in X) \quad P_{a+S} x = a + P_S(x - a). \quad (11)$$

Example 2.7. Let $n \geq 1$. Suppose³ that $X = \mathbb{R}^n$, that $p \in \mathbb{R}_{++}^n$ and that $T = p + P_{\mathbb{R}_+^n}$. Then T is (firmly) nonexpansive, $\text{Fix } T = \emptyset$, $\text{ran}(\text{Id} - T) = -p + \mathbb{R}_-^n$, $v_T = -p \in \text{ran}(\text{Id} - T)$ and $\text{int Fix}(v_T + T) = \mathbb{R}_-^n \neq \emptyset$. Consequently $\sum_{n=0}^{\infty} \|T^n x - T^{n+1} x - v_T\| < +\infty$ and $(T^n x + n v_T)_{n \in \mathbb{N}}$ converges.

Proof. The claim that T is firmly nonexpansive (hence nonexpansive) follows from e.g., [31, Section 3]. Now $\text{Id} - T = \text{Id} - p - P_{\mathbb{R}_+^n} = -p + P_{\mathbb{R}_-^n}$, hence $\text{ran}(\text{Id} - T) = -p + \mathbb{R}_-^n$ and $\text{Fix } T = \emptyset \Leftrightarrow 0 \notin \text{ran}(\text{Id} - T) = -p + \mathbb{R}_-^n \Leftrightarrow p \notin \mathbb{R}_-^n$, which is true. Using (11) with (a, S) replaced by $(-p, \mathbb{R}_-^n)$ we have $v_T = P_{-p+\mathbb{R}_-^n} 0 = -p + P_{\mathbb{R}_-^n} p = -p$. Consequently $v_T + T = -p + p + P_{\mathbb{R}_+^n} = P_{\mathbb{R}_+^n}$ and therefore $\text{Fix}(v_T + T) = \mathbb{R}_+^n$ which implies that $\text{int Fix}(v_T + T) = \mathbb{R}_{++}^n$. Now apply Proposition 2.6. ■

Corollary 2.8. Suppose that $X = \mathbb{R}$, that $\text{Fix } T = \emptyset$ and that $v_T := P_{\overline{\text{ran}}(\text{Id}-T)}0 \in \text{ran}(\text{Id} - T)$. Then $\text{int Fix}(v_T + T) \neq \emptyset$ and $\sum_{n=0}^{\infty} \|T^n x - T^{n+1} x - v_T\| < +\infty$. Consequently $(T^n x + n v_T)_{n \in \mathbb{N}}$ converges.

³Let $n \in \mathbb{N}$. The positive orthant in \mathbb{R}^n is $\mathbb{R}_+^n = [0, +\infty[^n$ and the strictly positive orthant in \mathbb{R}^n is $\mathbb{R}_{++}^n =]0, +\infty[^n$. Likewise we define the negative orthant and the strictly negative orthant \mathbb{R}_-^n and \mathbb{R}_{--}^n , respectively.

Proof. It follows from [6, Proposition 2.5(i)] that $\text{Fix}(v + T)$ contains an unbounded interval, and therefore, since $X = \mathbb{R}$, we conclude that $\text{int Fix}(v + T) \neq \emptyset$. Now apply Proposition 2.6. (See also [10, Theorem 3.6]). \blacksquare

3 The forward-backward operator and duality

The *primal* problem for the ordered pair (A, B) is

$$(P) \text{ find } x \in X \text{ such that } 0 \in Ax + Bx. \quad (12)$$

The Attouch-Théra *dual* pair [1] for the ordered pair (A, B) is the pair⁴ $(A^{-1}, B^{-\odot})$ and the corresponding *dual problem* is

$$(D) \text{ find } x \in X \text{ such that } 0 \in A^{-1}x + B^{-\odot}x. \quad (13)$$

The sets of primal and dual solutions for the ordered pair (A, B) , denoted respectively by Z and K are

$$Z := (A + B)^{-1}(0) \quad \text{and} \quad K := (A^{-1} + B^{-\odot})(0). \quad (14)$$

From now on we assume that

$$A : X \rightarrow X \text{ is firmly nonexpansive.} \quad (15)$$

The *forward-backward* algorithm to solve (12) iterates the operator

$$T_{\text{FB}} := T_{\text{FB}(A, B)} := J_B(\text{Id} - A). \quad (16)$$

On the other hand the *Douglas-Rachford* algorithm to solve (12) iterates the operator

$$T_{\text{DR}} := T_{\text{DR}(A, B)} := \text{Id} - J_A + J_B R_A. \quad (17)$$

Let $x \in X$. If $Z \neq \emptyset$ then each of the sequences $(T_{\text{FB}}^n x)_{n \in \mathbb{N}}$ (see, e.g., [23, Corollary 6.5] or [8, Section 25.3]) and $(J_A T_{\text{DR}}^n x)_{n \in \mathbb{N}}$ (see, e.g., [46] or [34]) converges weakly to a (possibly different) solution of (12).

Remark 3.1. Let $\alpha > 0$. Since $\text{zer}(A + B) = \text{zer}(\alpha A + \alpha B)$, the assumption that A is firmly nonexpansive could be replaced by A is α -cocoercive⁵. In this case (16) and (17) can be applied with the ordered pair (A, B) is replaced by $(\alpha A, \alpha B)$.

Definition 3.2 (paramonotone and 3* monotone operators). Let $C : X \rightrightarrows X$ be monotone. Then

⁴Let $B : X \rightrightarrows X$. Then $B^{\odot} := (-\text{Id}) \circ B \circ (-\text{Id})$ and $B^{-\odot} := (B^{-1})^{\odot} = (B^{\odot})^{-1}$ (see [7, Equation (10)]).

⁵Recall that $A : X \rightarrow X$ is cocoercive if $(\exists \alpha > 0)$ such that αA is firmly nonexpansive.

(i) C is paramonotone⁶ if $(\forall (x, u) \in \text{gra } C) (\forall (y, v) \in \text{gra } C)$ we have

$$\left. \begin{array}{l} (x, u) \in \text{gra } C \\ (y, v) \in \text{gra } C \\ \langle x - y, u - v \rangle = 0 \end{array} \right\} \Rightarrow \{(x, v), (y, u)\} \subseteq \text{gra } C. \quad (18)$$

(ii) C is 3^* monotone⁷ (this is also known as rectangular) if

$$(\forall x \in \text{dom } C)(\forall v \in \text{ran } C) \inf_{(z, w) \in \text{gra } C} \langle x - z, v - w \rangle > -\infty. \quad (19)$$

Lemma 3.3. *The following hold:*

- (i) A is maximally monotone.
- (ii) A is paramonotone.
- (iii) A is 3^* monotone.

Proof. (i): This is [8, Example 20.27]. (ii) & (iii): Note that $A = \text{Id} - (\text{Id} - A)$ and $\text{Id} - A$ is firmly nonexpansive. The conclusion follows from [14, Theorem 6.1]. ■

Proposition 3.4. *The following hold:*

- (i) T_{FB} is averaged.
- (ii) T_{FB} is asymptotically regular.
- (iii) K is a singleton.
- (iv) $Z = \text{Fix } T_{\text{FB}}$.
- (v) $K = A(Z) = A(\text{Fix } T_{\text{FB}})$.

Proof. (i): Since A is firmly nonexpansive so is $\text{Id} - A$ (see, e.g., [23, Lemma 2.3]). Note that J_B is firmly nonexpansive by [41, Proposition 1(c)]. It follows from [8, Remark 4.24(iii)] that $\text{Id} - A$ and J_B are $1/2$ -averaged and therefore $T = J_B(\text{Id} - A)$ is $2/3$ -averaged by [23, Lemma 2.2(iii)]. (ii): Combine (i) and Fact 2.3. (iii): Let k_1 and k_2 be in K . It follows from [7, Proposition 2.4] that $(\exists z_i \in Z)$ such that $k_i \in Az_i \cap (-Bz_i) = Az_i$, $i \in \{1, 2\}$. Since A is single-valued, we conclude that $k_i = Az_i$, $i \in \{1, 2\}$. Using [7, Corollary 2.13] we learn that $\langle z_1 - z_2, k_1 - k_2 \rangle = \langle z_1 - z_2, Az_1 - Az_2 \rangle = 0$. Now combine with Lemma 3.3(ii) and use that A is single-valued to learn that $k_1 = k_2$. (iv): This follows from [8, Proposition 25.1(iv)]. (v): In view of (iii), let $K = \{k\}$. It follows from [7] that $(\forall z \in Z) k = Az \cap (-Bz)$, which implies, since A is single-valued, that $k = Az$; equivalently $K = \{k\} = A(Z)$. Now combine with (iv). ■

Fact 3.5 (Baillon-Haddad). *Let $f: X \rightarrow \mathbb{R}$ be convex and differentiable. Then*

$$\nabla f \text{ is nonexpansive} \Leftrightarrow \nabla f \text{ is firmly nonexpansive.} \quad (20)$$

Proof. See [5, Corollaire 10]. ■

⁶For detailed discussion and examples of paramonotone operators we refer the reader to [30].

⁷For detailed discussion and examples of 3^* monotone operators we refer the reader to [18].

Fact 3.6. Let $f: X \rightarrow]-\infty, +\infty]$ be convex, lower semicontinuous, and proper. Then the following hold:

- (i) ∂f is maximally monotone.
- (ii) $(\partial f)^{-1} = \partial f^*$.

Proof. (i): See, e.g., [43, Theorem A]. (ii): See, e.g., [43, Remark on page 216], [29, Théorème 3.1], or [8, Corollary 16.24]. ■

Suppose that C is a nonempty closed convex subset of X . It is well-known (see, e.g., [8, Example 23.4]) that

$$J_{N_C} = P_C. \quad (21)$$

Proposition 3.7. Suppose that $f: X \rightarrow \mathbb{R}$ is convex and differentiable such that ∇f is nonexpansive and that $g: X \rightarrow]-\infty, +\infty]$ is convex, lower semicontinuous, and proper. Suppose that $A = \nabla f$ and that $B = \partial g$. Then the following hold^{8,9}:

- (i) $\text{Fix } T_{\text{FB}} = \text{zer}(\nabla f + \partial g) = \text{argmin}(f + g)$.
- (ii) $T_{\text{FB}} = \text{Prox}_g(\text{Id} - \nabla f)$.

If in addition, $g = \iota_V$ where V is a nonempty closed convex subset of X , then we have

$$(iii) \quad T_{\text{FB}} = P_V(\text{Id} - \nabla f).$$

Proof. Note that $\text{dom } f = X$ and that $A = \nabla f$ is firmly nonexpansive by Fact 3.5. (i): The first identity is Proposition 3.4(iv) applied with (A, B) replaced by $(\nabla f, \partial g)$. It follows from [22, Proposition 3.2 & Corollary 3.4] that $A + B = \nabla f + \partial g = \partial(f + g)$. Now apply [8, Proposition 26.1]. (ii): Combine (16) and [8, Example 23.3]. (iii): Combine (ii) and (21). ■

Remark 3.8. Let $f: X \rightarrow \mathbb{R}$ be convex and differentiable with $1/\beta$ Lipschitz continuous gradient, where $\beta > 0$. Then $\beta \nabla f$ is nonexpansive, hence firmly nonexpansive by Fact 3.5. Since $\text{argmin}(f + g) = \text{argmin}(\beta f + \beta g)$, Proposition 3.7 can be applied, with (f, g) replaced by $(\beta f, \beta g)$, to find a minimizer of $f + g$.

Suppose that¹⁰ C is a nonempty closed convex subset of X . In the sequel we make use of the following useful result (see, e.g., [37, Example on page 286] or [8, Corollary 12.30]).

$$\nabla \left(\frac{1}{2} d_C^2 \right) = \text{Id} - P_C. \quad (22)$$

Example 3.9 (Method of Alternating Projections (MAP) as a forward-backward iteration). Suppose that U and V are nonempty closed convex subsets of X , that $f = \frac{1}{2} d_U^2$ and that $g = \iota_V$. Suppose that $A = \nabla f = \text{Id} - P_U$ and that $B = \partial g = N_V$. Then A is firmly nonexpansive and

$$T_{\text{FB}(\text{Id} - P_U, N_V)} = P_V P_U. \quad (23)$$

⁸Let $h: X \rightarrow]-\infty, +\infty]$ be proper. The set of minimizers of h , $\{x \in X \mid h(x) = \inf h(X)\}$, is denoted by $\text{argmin } h$.

⁹Suppose that $g: X \rightarrow]-\infty, +\infty]$ is convex, lower semicontinuous, and proper. Then Prox_g is the Moreau prox operator associated with g defined by $\text{Prox}_g: X \rightarrow X: x \mapsto (\text{Id} + \partial g)^{-1}(x) = \text{argmin}_{y \in X} \left(g(y) + \frac{1}{2} \|x - y\|^2 \right)$.

¹⁰Let C be a nonempty closed convex subset of X . We use d_C to denote the distance from the set C defined by $d_C: X \rightarrow [0, +\infty[: x \mapsto \min_{c \in C} \|x - c\| = \|x - P_C x\|$.

Proof. It follows from (22) that $\nabla f = \text{Id} - P_U$, which is firmly nonexpansive by e.g., [48, Equation 1.7 on page 241]. Moreover (21) implies that $J_B = J_{N_V} = P_V$. Consequently, $T_{\text{FB}(A,B)} = J_B \circ (\text{Id} - A) = P_V(\text{Id} - (\text{Id} - P_U)) = P_V P_U$. ■

4 The forward-backward operator and the normal problem

Let $C : X \rightrightarrows X$ and let $w \in X$. The *inner shift* and *outer shift* of an operator C by w at $x \in X$ are defined by

$$C_w x := C(x - w) \quad \text{and} \quad {}_w C x := -w + Cx, \quad (24)$$

respectively.

Let $w \in X$. The w -perturbed problem introduced in [12] is:

$$(P_w) \quad \text{find } x \in X \text{ such that } 0 \in {}_w A x + B_w x = A x + B(x - w) - w, \quad (25)$$

and the corresponding set of zeros is

$$Z_w := \{x \in X \mid 0 \in 0 \in {}_w A x + B_w x\} = \{x \in X \mid w \in A x + B(x - w)\}. \quad (26)$$

Proposition 4.1. *Let $w \in X$. Then*

$$T_{\text{FB}(wA, B_w)} = -{}_w T_{\text{FB}} = w + T_{\text{FB}}, \quad (27)$$

and

$$Z_w = \text{Fix}(w + T_{\text{FB}}). \quad (28)$$

Moreover, the following are equivalent:

- (i) $Z_w \neq \emptyset$.
- (ii) $w \in \text{ran}(A + B_w)$.
- (iii) $w \in \text{ran}(\text{Id} - T_{\text{FB}})$.
- (iv) $w \in \text{ran}(\text{Id} - T_{\text{DR}})$.

Proof. Let $x \in X$. Using (16) and [8, Proposition 23.15(ii)&(iii)] we have $T_{\text{FB}(wA, B_w)} x = J_{B_w}(\text{Id} - {}_w A)x = J_B((x - (Ax - w)) - w) + w = J_B(x - Ax) + w = J_B(\text{Id} - A)x + w = w + T_{\text{FB}}x$, which proves (27). To prove (28) apply Proposition 3.4(iv) with (A, B) replaced by $({}_w A, B_w)$ and use (27). “(i) \Leftrightarrow (ii)”: This follows from (26). “(i) \Leftrightarrow (iii)”: Indeed, using (28) we have $Z_w \neq \emptyset \Leftrightarrow \text{Fix}(w + T_{\text{FB}}) \neq \emptyset \Leftrightarrow (\exists x \in X) \text{ such that } x = w + T_{\text{FB}}x \Leftrightarrow w \in \text{ran}(\text{Id} - T_{\text{FB}})$. “(i) \Leftrightarrow (iv)”: This follows from [12, Proposition 3.3]. ■

Theorem 4.2. *We have*¹¹

- (i) $\text{ran}(\text{Id} - T_{\text{DR}}) = \text{ran}(\text{Id} - T_{\text{FB}})$.

¹¹For convenience we shall use v_{FB} and v_{DR} to denote $v_{T_{\text{FB}}}$ and $v_{T_{\text{DR}}}$ respectively.

- (ii) $\text{ran}(\text{Id} - T_{\text{FB}}) \subseteq \text{ran } A + \text{ran } B$.
- (iii) $v_{\text{DR}} = v_{\text{FB}}$.

Proof. (i): This is clear from the equivalence of (iii) and (iv) in Proposition 4.1. (ii): Combine (i) and [28, Proposition 4.1]. (iii): Indeed, using (i) and (6) we have $v_{\text{DR}} = P_{\overline{\text{ran}}(\text{Id} - T_{\text{DR}})}0 = P_{\overline{\text{ran}}(\text{Id} - T_{\text{FB}})}0 = v_{\text{FB}}$. ■

In view of Theorem 4.2(i), it is tempting to ask whether we can derive a similar conclusion for the equality of $\text{ran } T_{\text{FB}}$ and $\text{ran } T_{\text{DR}}$. The next example gives a negative answer to this conjecture.

Example 4.3 ($\text{ran } T_{\text{DR}} \neq \text{ran } T_{\text{FB}}$). Suppose that $A = \text{Id}$. Then $T_{\text{DR}} = \frac{1}{2}\text{Id} + J_B 0$ and $T_{\text{FB}} \equiv J_B 0$. Consequently,

$$X = \text{ran } T_{\text{DR}} \neq \text{ran } T_{\text{FB}} = \{J_B 0\}. \quad (29)$$

Proof. One can easily verify that $J_A = \frac{1}{2}\text{Id}$, hence $R_A \equiv 0$. Therefore,

$$T_{\text{DR}} = \text{Id} - J_A + J_B R_A = \text{Id} - \frac{1}{2}\text{Id} + J_B 0 = \frac{1}{2}\text{Id} + J_B 0, \quad (30)$$

and

$$T_{\text{FB}} = J_B(\text{Id} - A) = J_B(\text{Id} - \text{Id}) \equiv J_B 0, \quad (31)$$

and the conclusion readily follows. ■

Unlike the Douglas–Rachford operator, where we can learn about $\text{ran } T_{\text{DR}}$ (see [11, Corollary 5.3]), we cannot obtain accurate information about the range of T_{FB} as we show next.

Lemma 4.4. $\text{ran } T_{\text{FB}} \subseteq \text{dom } B$.

Proof. Indeed, $\text{ran } T_{\text{FB}} \subseteq \text{ran } J_B = \text{ran}(\text{Id} + B)^{-1} = \text{dom}(\text{Id} + B) = \text{dom } B$. ■

The result in Lemma 4.4, cannot be improved as we illustrate now.

Example 4.5 ($\text{ran } T_{\text{FB}} \subsetneq \text{dom } B$). Suppose that $A = \text{Id}$ and that $\text{dom } B$ is not a singleton. Then Example 4.3 implies that $\{J_B 0\} = \text{ran } T_{\text{FB}} \subsetneq \text{dom } B$.

Example 4.6 ($\text{ran } T_{\text{FB}} = \text{dom } B$). Let C be a nonempty closed convex subset of X . Suppose that $A \equiv 0$ and that $B = N_C$. Then (21) implies that $T_{\text{FB}} = J_B = P_C$, hence $\text{ran } T_{\text{FB}} = C = \text{dom } B$.

The *normal problem* (see [12, Definition 3.7]) associated with the ordered pair (A, B) is the v -perturbed problem where v is the *minimal displacement vector* defined by

$$v := v_{\text{FB}} := P_{\overline{\text{ran}}(\text{Id} - T_{\text{FB}})}0; \quad (32)$$

and the corresponding set of *normal solutions* is Z_v .

Corollary 4.7. $Z_v = \text{Fix}(v + T_{\text{FB}})$.

Proof. This follows from Proposition 4.1. ■

We point out that, even though the normal problem is well-defined in view of Fact 2.4, the set of normal solution may or may not be empty, as we illustrate now.

Example 4.8 ($Z = \emptyset$ but normal solutions exist). *Let $a^*, b^* \in X$ such that $a^* + b^* \neq 0$. Suppose that $A: X \rightarrow X: x \mapsto a^*$, and that $B: X \rightarrow X: x \mapsto b^*$. Then $Z = \emptyset$, $\text{ran}(\text{Id} - T_{\text{FB}}) = \{a^* + b^*\}$, therefore $v = a^* + b^* \in \text{ran}(\text{Id} - T_{\text{FB}})$ and $Z_v = X \neq \emptyset$.*

Proof. We have $J_B = (\text{Id} + b^*)^{-1} = \text{Id} - b^*$, and $T_{\text{FB}} = J_B(\text{Id} - A) = \text{Id} - (a^* + b^*)$. Consequently, $\overline{\text{ran}}(\text{Id} - T_{\text{FB}}) = \text{ran}(\text{Id} - T_{\text{FB}}) = \{a^* + b^*\}$ and $v = a^* + b^* \in \text{ran}(\text{Id} - T_{\text{FB}})$. Therefore, $(\forall x \in X) x - T_{\text{FB}}x = a^* + b^* = v$, which in view of Corollary 4.7, implies that $Z_v = X$, as claimed. ■

Example 4.9 ($Z = \emptyset$ and normal solutions do not exist). *Suppose that $X = \mathbb{R}^2$, that $U = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y \geq 1/x\}$, that $V = \mathbb{R} \times \{0\}$, that $\beta < 0$, that $w = (\beta, 0) \neq (0, 0)$ and that $f = \frac{1}{2}d_U^2$. Set $A = \nabla f$ and set $B = w + N_V$. Then $T_{\text{FB}} = -w + P_V P_U$, $v = w$, $v \notin \text{ran}(\text{Id} - T_{\text{FB}})$ and therefore $Z_v = \emptyset$.*

Proof. In view of (22) we have $A = \text{Id} - P_U$. Moreover (21) and [8, Proposition 23.15(ii)] implies that $J_B = P_V(\cdot - w) = P_V - w$, where the last identity uses that P_V is linear and that $w \in V$. Consequently $T_{\text{FB}} = J_B(\text{Id} - A) = P_V(\text{Id} - (\text{Id} - P_U)) - w = P_V P_U - w$. We claim that

$$\text{ran}(\text{Id} - T_{\text{FB}}) = w + \text{ran}(\text{Id} - P_V P_U). \quad (33)$$

Indeed, let $y \in X$. Then $y \in \text{ran}(\text{Id} - T_{\text{FB}}) \Leftrightarrow (\exists x \in X)$ such that $y = w + x - P_V P_U x \Leftrightarrow y \in w + \text{ran}(\text{Id} - P_V P_U)$. It follows from Example 5.8 below that $\overline{\text{ran}}(\text{Id} - P_V P_U) = (\text{rec } U)^\ominus + (\text{rec } V)^\ominus = \mathbb{R}_-^2 + V^\perp = \mathbb{R}_-^2 + (\{0\} \times \mathbb{R}) = \mathbb{R}_- \times \mathbb{R}$. Using (11) applied with S replaced by $\text{ran}(\text{Id} - P_V P_U)$ we have $v = w + P_{\overline{\text{ran}}(\text{Id} - P_V P_U)}(-w) = w$. Consequently (33) becomes $\text{ran}(\text{Id} - T_{\text{FB}}) = v + \text{ran}(\text{Id} - P_V P_U)$. Furthermore, using [13, Lemma 2.2(i)] $v \in \text{ran}(\text{Id} - T_{\text{FB}}) = v + \text{ran}(\text{Id} - P_V P_U) \Leftrightarrow 0 \in \text{ran}(\text{Id} - P_V P_U) \Leftrightarrow \text{Fix } P_V P_U \neq \emptyset \Leftrightarrow U \cap V \neq \emptyset$, which does not hold, hence $Z_v = \emptyset$ by Proposition 4.1. ■

Remark 4.10. *Suppose that A^{-1} is firmly nonexpansive. Then one can define the forward-backward operator for the dual pair $(A^{-1}, B^{-\ominus})$. Nonetheless, the self-duality property, which is a key feature of T_{DR} (see, e.g., [7, Corollary 4.3] or [28, Lemma 3.6 on page 133]), does not hold for T_{FB} as we illustrate in Example 4.11.*

Example 4.11 (T_{FB} is not self-dual). *Suppose that V is a closed linear subspace of X and let $u \in V \setminus \{0\}$. Suppose that $A: X \rightarrow X: x \mapsto x - u$ and that $B = N_V$. Then A^{-1} is firmly nonexpansive, however*

$$u \equiv T_{\text{FB}(A, B)} \neq T_{\text{FB}(A^{-1}, B^{-\ominus})} \equiv 0. \quad (34)$$

Proof. First note that $A^{-1}: X \rightarrow X: x \mapsto x + u$, hence A^{-1} is firmly nonexpansive, as claimed. Since B is linear we learn that B^{-1} is linear and so are J_B and $J_{B^{-1}}$ by [15, Theorem 2.1(xviii)]. By [7, Proposition 4.1(ii)] and Fact 2.1(ii) we have $J_{B^{-\ominus}} = J_{(B^{-1})^\ominus} = J_{B^{-1}} = \text{Id} - J_B = \text{Id} - P_V = P_{V^\perp}$. Now, $T_{\text{FB}(A, B)} = J_B(\text{Id} - A) = P_V(\text{Id} - \text{Id} + u) = P_V u = u$, whereas $T_{\text{FB}(A^{-1}, B^{-\ominus})} = J_{B^{-\ominus}}(\text{Id} - A^{-1}) = P_{V^\perp}(\text{Id} - \text{Id} - u) = P_{V^\perp}(-u) = -P_{V^\perp}(u) \equiv 0$. ■

Remark 4.12. Clearly the forward-backward operator is not symmetric in A and B , however, it is critical to consider the order in (16) when only A is firmly nonexpansive. If, in addition, B is firmly nonexpansive we can also define $T_{\text{FB}(B,A)}$.

Corollary 4.13. Suppose that $B : X \rightarrow X$ is firmly nonexpansive. Then $T_{\text{FB}(B,A)} := J_A(\text{Id} - B)$ is averaged and

$$\|v_{\text{FB}(A,B)}\| = \|v_{\text{FB}(B,A)}\|. \quad (35)$$

Proof. Combining Theorem 4.2(iii) and [12, Proposition 3.11] we have $\|v_{\text{FB}(A,B)}\| = \|v_{\text{DR}(A,B)}\| = \|v_{\text{DR}(B,A)}\| = \|v_{\text{FB}(B,A)}\|$. ■

5 The range of the displacement operator

Unless otherwise stated, in this section we work under the assumption that

H is a finite-dimensional Hilbert space.

The results in this section provide information on the range of the displacement map $\text{Id} - T_{\text{FB}}$.

Definition 5.1 (nearly convex and nearly equal sets). Let C and D be subsets¹² of H .

- (i) We say that D is nearly convex¹³ (see [40, Theorem 12.41]) if there exists a convex set subset E of H such that $E \subseteq D \subseteq \overline{E}$.
- (ii) We say that C and D are nearly equal¹⁴ if

$$C \simeq D \Leftrightarrow \overline{C} = \overline{D} \text{ and } \text{ri } C = \text{ri } D. \quad (36)$$

Fact 5.2. Let H be a finite-dimensional Hilbert space. Let $C : H \rightrightarrows H$ be maximally monotone. Then $\text{dom } C$ and $\text{ran } C$ are nearly convex.

Proof. See [40, Theorem 12.41]. ■

Theorem 5.3. Let H be a finite-dimensional Hilbert space. The following hold:

- (i) $\text{ran}(\text{Id} - T_{\text{FB}}) \simeq \text{ran } A + \text{ran } B$.
- (ii) Suppose that A and B are affine¹⁵. Then $\text{ran}(\text{Id} - T_{\text{FB}}) = \overline{\text{ran}}(\text{Id} - T_{\text{FB}}) = \text{ran } A + \text{ran } B$.

If, in addition, A or B is surjective then we additionally have:

- (iii) $\text{ran}(\text{Id} - T_{\text{FB}}) = X$.
- (iv) $\text{Fix } T_{\text{FB}} = Z \neq \emptyset$.

¹²Let C be a subset of H . We use $\text{ri } C$ to denote the interior of C with respect to the affine hull of C .

¹³For detailed discussion on the algebra of nearly convex sets we refer the reader to [42, Section 3].

¹⁴For detailed discussion on the properties of nearly equal and nearly convex sets we refer the reader to [15].

¹⁵Let $B : X \rightrightarrows X$. Then B is an affine relation if $\text{gra } B$ is an affine subspace of $X \times X$.

Proof. (i): Note that A is 3^* monotone (by Lemma 3.3(iii)) and $\text{dom } A = X$. It follows from [11, Theorem 5.2] that $\text{ran}(\text{Id} - T_{\text{DR}}) \simeq (\text{dom } A - \text{dom } B) \cap (\text{ran } A + \text{ran } B)$. Now combine with Theorem 4.2(i) and use that $\text{dom } A = X$. (ii): On the one hand, $\text{ran } A$ and $\text{ran } B$ are closed affine subspaces of X , so is their sum $\text{ran } A + \text{ran } B$. On the other hand, since the resolvent J_B is affine (see [15, Theorem 2.1(xix)]), so are T_{FB} and $\text{Id} - T_{\text{FB}}$. Therefore, in view of (i), $\text{ran}(\text{Id} - T_{\text{FB}}) = \overline{\text{ran}(\text{Id} - T_{\text{FB}})} = \overline{\text{ran } A + \text{ran } B} = \text{ran } A + \text{ran } B$. (iii): Using Theorem 5.3(i) we have $X = \text{ri } X = \text{ri}(\text{ran } A + \text{ran } B) \subseteq \text{ran}(\text{Id} - T_{\text{FB}}) \subseteq \overline{\text{ran}(\text{Id} - T_{\text{FB}})} = \overline{\text{ran } A + \text{ran } B} = X$. (iv): Note that in view of Proposition 3.4(iv) $0 \in \text{ran}(\text{Id} - T_{\text{FB}}) \Leftrightarrow \text{Fix } T_{\text{FB}} \neq \emptyset \Leftrightarrow Z \neq \emptyset$. Now combine with (iii). ■

In the conclusion of Theorem 5.3(i), we cannot replace near equality by equality as we illustrate in Example 5.4.

Example 5.4. Suppose that $H = \mathbb{R}^2$ and let $f: \mathbb{R}^2 \rightarrow]-\infty, +\infty] : (\xi_1, \xi_2) \mapsto \max \{g(\xi_1), |\xi_2|\}$, where $g(\xi_1) = 1 - \sqrt{\xi_1}$ if $\xi_1 \geq 0$ and $g(\xi_1) = +\infty$ otherwise. Set¹⁶ $A = P_{\mathbb{R}_+^2}$ and $B = \partial f^*$. Then A is firmly nonexpansive and B is maximally monotone. Moreover, $\text{ran } A = \mathbb{R}_+^2$, $\text{ran } B = \{(\xi_1, \xi_2) \mid \xi_1 > 0, \xi_2 \in \mathbb{R}\} \cup \{(0, \xi_2) \mid |\xi_2| \geq 1\}$, hence $\text{ran } A + \text{ran } B = \{(\xi_1, \xi_2) \mid \xi_1 \geq 0, \xi_2 \in \mathbb{R}\}$ but $\text{ran}(\text{Id} - T_{\text{FB}}) = \{(\xi_1, \xi_2) \mid \xi_1 > 0, \xi_2 \in \mathbb{R}\} \cup \{(0, \xi_2) \mid \xi_2 \leq -1\}$. Therefore

$$\text{ri}(\text{ran } A + \text{ran } B) \subsetneq \text{ran}(\text{Id} - T) \subsetneq \overline{\text{ran } A + \text{ran } B} = \text{ran } A + \text{ran } B. \quad (37)$$

Proof. The claim about firm nonexpansiveness of A follows from e.g., [48, Equation 1.6 on page 241] or [31, Section 3] and maximal monotonicity of B follows from Fact 3.6(i) applied to f^* . Using Fact 3.6(ii) and [42, Example on page 218] we see that $\text{dom } \partial f = \text{ran}(\partial f)^{-1} = \text{ran } \partial f^* = \text{ran } B = \{(\xi_1, \xi_2) \mid \xi_1 > 0, \xi_2 \in \mathbb{R}\} \cup \{(0, \xi_2) \mid |\xi_2| \geq 1\}$. Note that in view of Theorem 5.3(i) we have $\{(\xi_1, \xi_2) \mid \xi_1 > 0, \xi_2 \in \mathbb{R}\} = \text{ri}(\text{ran } A + \text{ran } B) \subseteq \text{ran}(\text{Id} - T) \subseteq \overline{\text{ran } A + \text{ran } B} = \{(\xi_1, \xi_2) \mid \xi_1 \geq 0, \xi_2 \in \mathbb{R}\}$. Therefore we only need to check the points in $\{(0, \beta) \mid \beta \in \mathbb{R}\}$. To

¹⁶Let $f: X \rightarrow]-\infty, +\infty]$ be convex, lower semicontinuous, and proper. We use f^* to denote the *convex conjugate* (a.k.a. *Fenchel conjugate*) of f , defined by $f^*: X \rightarrow]-\infty, +\infty] : x \mapsto \sup_{u \in X} (\langle x, u \rangle - f(u))$.

proceed further we recall that (see [36, Example 6.5])

$$\partial f(\xi_1, \xi_2) = \begin{cases} \emptyset, & \text{if } \xi_1 < 0; \\ \emptyset, & \text{if } \xi_1 = 0 \text{ and } |\xi_2| < 1; \\ \mathbb{R}_- \times \{1\}, & \text{if } \xi_1 = 0 \text{ and } \xi_2 \geq 1; \\ \mathbb{R}_- \times \{-1\}, & \text{if } \xi_1 = 0 \text{ and } \xi_2 \leq -1; \\ \text{conv} \left\{ \left(-\frac{1}{2}\xi_1^{-1/2}, 0\right), (0, 1) \right\}, & \text{if } \xi_2 = 1 - \sqrt{\xi_1} \text{ and } 0 < \xi_1 < 1; \\ \text{conv} \left\{ \left(-\frac{1}{2}\xi_1^{-1/2}, 0\right), (0, -1) \right\}, & \text{if } -\xi_2 = 1 - \sqrt{\xi_1} \text{ and } 0 < \xi_1 < 1; \\ \left(-\frac{1}{2}\xi_1^{-1/2}, 0\right), & \text{if } 0 < \xi_1 < 1 \text{ and } 1 - \sqrt{\xi_1} > |\xi_2|; \\ (0, 1), & \text{if } 0 < \xi_1 < 1 \text{ and } \xi_2 > 1 - \sqrt{\xi_1}; \\ (0, -1), & \text{if } 0 < \xi_1 < 1 \text{ and } -\xi_2 > 1 - \sqrt{\xi_1}; \\ \text{conv} \left\{ \left(-\frac{1}{2}, 0\right), (0, 1), (0, -1) \right\}, & \text{if } \xi_1 = 1 \text{ and } \xi_2 = 0; \\ \text{conv} \left\{ (0, 1), (0, -1) \right\}, & \text{if } \xi_1 > 1 \text{ and } \xi_2 = 0; \\ (0, 1), & \text{if } \xi_1 > 1 \text{ and } \xi_2 > 0; \\ (0, -1), & \text{if } \xi_1 > 1 \text{ and } -\xi_2 > 0. \end{cases} \quad (38)$$

Let $\beta \in \mathbb{R}$. In view of Proposition 4.1 and Fact 3.6(ii) we have

$$(0, \beta) \in \text{ran}(\text{Id} - T_{\text{FB}}) \Leftrightarrow (\exists (\xi_1, \xi_2) \in \mathbb{R}^2) (0, \beta) \in P_{\mathbb{R}_+^2}(\xi_1, \xi_2) + \partial f^*(\xi_1, \xi_2 - \beta) \quad (39a)$$

$$= P_{\mathbb{R}_+^2}(\xi_1, \xi_2) + (\partial f)^{-1}(\xi_1, \xi_2 - \beta) \quad (39b)$$

$$\Leftrightarrow (\exists (\xi_1, \xi_2) \in \mathbb{R}^2) (0, \beta) - P_{\mathbb{R}_+^2}(\xi_1, \xi_2) \in (\partial f)^{-1}(\xi_1, \xi_2 - \beta) \quad (39c)$$

$$\Leftrightarrow (\exists (\xi_1, \xi_2) \in \mathbb{R}^2) (\xi_1, \xi_2 - \beta) \in \partial f \left((0, \beta) - P_{\mathbb{R}_+^2}(\xi_1, \xi_2) \right). \quad (39d)$$

We argue by cases using (38) and (39).

Case 1: $\xi_1 \geq 0$ and $\xi_2 \geq 0$. Then $(0, \beta) \in \text{ran}(\text{Id} - T_{\text{FB}}) \Leftrightarrow (\exists (\xi_1, \xi_2) \in \mathbb{R}^2) (\xi_1, \xi_2 - \beta) \in \partial f((0, \beta) - P_{\mathbb{R}_+^2}(\xi_1, \xi_2)) = \partial f(-\xi_1, \beta - \xi_2) \Leftrightarrow [(\exists (\xi_1, \xi_2) \in \mathbb{R}^2) \xi_1 = 0, \xi_2 - \beta = 1 \text{ and } \beta - \xi_2 \geq 1 \text{ or } \xi_1 = 0, \xi_2 - \beta = -1 \text{ and } \beta - \xi_2 \leq -1]$, which is impossible.

Case 2: $\xi_1 \leq 0$ and $\xi_2 \leq 0$. Then $(0, \beta) \in \text{ran}(\text{Id} - T_{\text{FB}}) \Leftrightarrow (\exists (\xi_1, \xi_2) \in \mathbb{R}^2) (\xi_1, \xi_2 - \beta) \in \partial f((0, \beta) - P_{\mathbb{R}_+^2}(\xi_1, \xi_2)) = \partial f(0, \beta) \Leftrightarrow [(\exists (\xi_1, \xi_2) \in \mathbb{R}^2) \xi_1 \leq 0, \xi_2 - \beta = 1 \text{ and } \beta \geq 1 \text{ or } \xi_1 \leq 0, \xi_2 - \beta = -1 \text{ and } \beta \leq -1] \Leftrightarrow [(\exists (\xi_1, \xi_2) \in \mathbb{R}^2) \xi_1 \leq 0, \xi_2 = \beta + 1 \geq 2 \text{ or } \xi_1 \leq 0, \xi_2 = \beta - 1 \leq -2]$. Since $\xi_2 \leq 0$ we conclude that $\beta \leq -1$.

Case 3: $\xi_1 > 0$ and $\xi_2 < 0$. Then $(0, \beta) \in \text{ran}(\text{Id} - T_{\text{FB}}) \Leftrightarrow (\exists (\xi_1, \xi_2) \in \mathbb{R}^2) (\xi_1, \xi_2 - \beta) \in \partial f((0, \beta) - P_{\mathbb{R}_+^2}(\xi_1, \xi_2)) = \partial f(-\xi_1, \beta) \Rightarrow [\xi_1 > 0 \text{ and by (38) } -\xi_1 > 0]$ which is impossible.

Case 4: $\xi_1 < 0$ and $\xi_2 > 0$. Then $(0, \beta) \in \text{ran}(\text{Id} - T_{\text{FB}}) \Leftrightarrow (\exists (\xi_1, \xi_2) \in \mathbb{R}^2) (\xi_1, \xi_2 - \beta) \in \partial f((0, \beta) - P_{\mathbb{R}_+^2}(\xi_1, \xi_2)) = \partial f(0, \beta - \xi_2) \Leftrightarrow [\xi_1 < 0, \xi_2 - \beta = 1 \text{ and } \beta - \xi_2 \geq 1 \text{ or } \xi_1 < 0, \xi_2 - \beta = -1 \text{ and } \beta - \xi_2 \leq -1]$, which never occurs.

Altogether we conclude that $\text{ran}(\text{Id} - T_{\text{FB}}) = \{(\xi_1, \xi_2) \mid \xi_1 > 0, \xi_2 \in \mathbb{R}\} \cup \{(0, \xi_2) \mid \xi_2 \leq -1\}$, as claimed. ■

Suppose that C and D are nonempty nearly convex subsets of H . Then [15, Proposition 2.12] implies that

$$C \simeq D \Leftrightarrow \overline{C} = \overline{D}. \quad (40)$$

Lemma 5.5. *Let H be a finite-dimensional Hilbert space. Suppose that $f: H \rightarrow]-\infty, +\infty]$ is convex, lower semicontinuous, and proper. Then the $\text{dom } \partial f \simeq \text{dom } f$ and $\text{ran } \partial f \simeq \text{dom } f^*$.*

Proof. It follows from Fact 5.2 and Fact 3.6(i) that $\text{dom } \partial f$ is nearly convex. Moreover, [8, Corollary 16.29] implies that $\overline{\text{dom } \partial f} = \overline{\text{dom } f}$. Therefore (40) implies that $\text{dom } \partial f \simeq \text{dom } f$. Using Fact 3.6(ii) we have $\text{ran } \partial f = \text{dom}(\partial f)^{-1} = \text{dom } \partial f^*$. Now apply the same argument to f^* . ■

We recall that (see [48, Theorem 3.1]) for a nonempty closed convex subset C of X the following holds¹⁷:

$$\overline{\text{ran}}(\text{Id} - P_C) = (\text{rec } C)^\ominus. \quad (41)$$

Example 5.6. *Let H be a finite-dimensional Hilbert space. Suppose that C is a nonempty closed convex subset of H . Set $f = \iota_C$ and suppose that $A = \partial f = N_C$. Then $\text{dom } A = C$ and $\text{ran } A \simeq (\text{rec } C)^\ominus$.*

Proof. Clearly $\text{dom } A = C$. It follows from [8, Proposition 23.2(i)], Fact 2.1(ii) and (21) that $\text{ran } A = \text{dom } A^{-1} = \text{ran } J_{A^{-1}} = \text{ran}(\text{Id} - J_A) = \text{ran}(\text{Id} - P_C)$. In view of (41) we have $\overline{\text{ran}}(\text{Id} - P_C) = (\text{rec } C)^\ominus$. Note that $J_{A^{-1}} = \text{Id} - P_C$ is maximally monotone by Fact 2.1(ii)&(i), therefore Fact 5.2 implies that $\text{ran}(\text{Id} - P_C)$ is nearly convex. Now apply (40). ■

Suppose that C_1 and C_2 are nearly convex subsets of H and that D_1 and D_2 are subsets of H such that $C_i \simeq D_i$ for every $i \in \{1, 2\}$. It follows from [15, Theorem 2.14] that

$$C_1 + C_2 \simeq D_1 + D_2. \quad (42)$$

Proposition 5.7. *Let H be a finite-dimensional Hilbert space. Suppose that $f: H \rightarrow \mathbb{R}$ is convex and differentiable such that ∇f is nonexpansive and that $g: H \rightarrow]-\infty, +\infty]$ is convex, lower semicontinuous, and proper. Suppose that $A = \nabla f$ and that $B = \partial g$. Then the following hold:*

$$(i) \text{ran}(\text{Id} - T_{\text{FB}}) \simeq \text{dom } f^* + \text{dom } g^*.$$

If in addition, $g = \iota_V$ where V is a nonempty closed convex subset of H , then we have:

$$(ii) \text{ran}(\text{Id} - T_{\text{FB}}) \simeq \text{dom } f^* + (\text{rec } V)^\ominus.$$

Proof. It follows from Fact 3.5 that ∇f is firmly nonexpansive. (i): Combine Theorem 5.3(i), Lemma 5.5 and (42). (ii): It follows from Lemma 5.5 and Example 5.6 respectively that $\text{ran } A \simeq \text{dom } f^*$ and $\text{ran } B \simeq (\text{rec } V)^\ominus$. Now combine with Theorem 5.3(i) and (42). ■

¹⁷Let C be a nonempty closed convex subset of X . The *recession cone* of C is $\text{rec } C := \{x \in X \mid x + C \subseteq C\}$, and the *polar cone* of C is $C^\ominus := \{u \in X \mid \sup_{c \in C} \langle c, u \rangle \leq 0\}$,

Example 5.8 (range of the displacement map of alternating projections). Let H be a finite-dimensional Hilbert space. Suppose that U and V are nonempty closed convex subsets of X , that $f = \frac{1}{2}d_U^2$ and that $g = \iota_V$. Suppose that $A = \nabla f = \text{Id} - P_U$ and that $B = \partial g = N_V$. Then

$$\text{ran}(\text{Id} - T_{\text{FB}}) = \text{ran}(\text{Id} - P_V P_U) \simeq (\text{rec } U)^\ominus + (\text{rec } V)^\ominus. \quad (43)$$

Proof. It follows from (41) and (40) that $\text{ran } A = \text{ran}(\text{Id} - P_U) \simeq (\text{rec } U)^\ominus$. On the other hand Example 5.6 implies that $\text{ran } B \simeq (\text{rec } V)^\ominus$. Now combine with [15, Theorem 2.12]. ■

6 Affine operators and applications

Fact 6.1. Let $L: X \rightarrow X$ be linear and nonexpansive, let $b \in X$ and suppose that $T: X \rightarrow X: x \mapsto Lx + b$. Let $v_T := P_{\overline{\text{ran}(\text{Id} - T)}}0$ and let $x \in X$. Then

$$(\forall n \in \mathbb{N}) \quad T^n x + n v_T = (T_{-v_T})^n x = (v_T + T)^n x. \quad (44)$$

Proof. See [6, Theorem 3.2(iv) and (v)]. ■

Lemma 6.2. Let $L: X \rightarrow X$ be linear and nonexpansive, let $b \in X$, suppose that $T: X \rightarrow X: x \mapsto Lx + b$ and that $v_T := P_{\overline{\text{ran}(\text{Id} - T)}}0 \in \text{ran}(\text{Id} - T)$. Let $x \in X$. Then there exists a point $a \in X$ such that $v_T + b = a - La$ and $v_T + Tx = a + L(x - a)$. Moreover we have

$$(\forall n \in \mathbb{N}) \quad T^n x + n v_T = (T_{-v_T})^n x = (v_T + T)^n x = a + L^n(x - a) \quad (45)$$

and

$$\text{Fix}(v_T + T) = a + \text{Fix } L. \quad (46)$$

Proof. Note that $v_T \in \text{ran}(\text{Id} - T) = \text{ran}(\text{Id} - L) - b \Leftrightarrow v_T + b \in \text{ran}(\text{Id} - L)$. Now let $a \in X$ be such that $v_T + b = a - La$. The first two identities in (45) follow from Fact 6.1. We prove the last identity in (45) by induction. The case $n = 0$ is obvious. Now suppose that for some $n \in \mathbb{N}$ $(v_T + T)^n x = a + L^n(x - a)$. Then $(v_T + T)^{n+1} x = v_T + b + L(a + L^n(x - a)) = v_T + b + La + L^{n+1}(x - a) = a + L^{n+1}(x - a)$. We now turn to (46). In view of (45) applied with $n = 1$ we have $x \in \text{Fix}(v_T + T) \Leftrightarrow x = v_T + Tx \Leftrightarrow x = a + L(x - a) \Leftrightarrow x - a \in \text{Fix } L \Leftrightarrow x \in a + \text{Fix } L$, hence $\text{Fix}(v_T + T) = a + \text{Fix } L$. ■

Proposition 6.3. Let $L: X \rightarrow X$ be linear and nonexpansive, let $b \in X$, suppose that $T: X \rightarrow X: x \mapsto Lx + b$ and that $v_T := P_{\overline{\text{ran}(\text{Id} - T)}}0 \in \text{ran}(\text{Id} - T)$. Let $x \in X$. Then $\text{Fix}(v_T + T) \neq \emptyset$. Moreover the following are equivalent:

- (i) L is asymptotically regular.
- (ii) $L^n x \rightarrow P_{\text{Fix } L} x$.
- (iii) $T^n x + n v_T = (v_T + T)^n x = (T_{-v_T})^n x \rightarrow P_{\text{Fix}(v_T + T)} x$.
- (iv) $T_{-v_T} = v_T + T$ is asymptotically regular.

(v) $(T^n x + nv_T)_{n \in \mathbb{N}}$ is asymptotically regular.

Proof. The proof uses the same techniques as in [16]. “(i) \Leftrightarrow (ii)”: See [4, Proposition 4], [3, Theorem 1.1], [9, Theorem 2.2] or [8, Proposition 5.27]. “(ii) \Rightarrow (iii)”: Using (45) and (11) we learn that

$$T^n x + nv_T = (T_{-v_T})^n x = (v_T + T)^n x = a + L^n(x - a) \quad (47a)$$

$$\rightarrow a + P_{\text{Fix } L}(x - a) = P_{a + \text{Fix } L} x = P_{\text{Fix}(v_T + T)} x. \quad (47b)$$

Now combine with (46). “(iii) \Rightarrow (iv)”: Clear. “(iv) \Rightarrow (v)”: This follows from Fact 6.1. “(v) \Rightarrow (i)”: Using (45) we have $L^n x - L^{n+1} x = T^n(x + a) + nv_T - (T^{n+1}(x + a) + (n + 1)v_T) \rightarrow 0$. ■

Let $\mathcal{B}(X)$ denote the set of bounded linear operators on X . We have the following result.

Proposition 6.4. *Let $L: X \rightarrow X$ be linear and nonexpansive, let $b \in X$, suppose that $T: X \rightarrow X: x \mapsto Lx + b$ and that $v_T := P_{\overline{\text{ran}(\text{Id} - T)}} 0 \in \text{ran}(\text{Id} - T)$. Let $x \in X$ and let $\mu \in]0, 1[$. Then the following are equivalent:*

- (i) $T^n x + nv_T = (v_T + T)^n x = (T_{-v_T})^n x \rightarrow P_{\text{Fix}(v_T + T)} x$ μ -linearly.
- (ii) $L^n x \rightarrow P_{\text{Fix } L} x$ μ -linearly.
- (iii) $L^n \rightarrow P_{\text{Fix } L}$ μ -linearly (in $\mathcal{B}(X)$).

Proof. Note that L is asymptotically regular by Fact 2.3. “(i) \Leftrightarrow (ii)”: In view of (45), (46) and (11) we learn that $T^n x + nv_T - P_{\text{Fix}(v_T + T)} x = (v_T + T)^n x - P_{\text{Fix}(v_T + T)} x = (T_{-v_T})^n x - P_{\text{Fix}(v_T + T)} x = a + L^n(x - a) - P_{a + \text{Fix } L} x = a + L^n(x - a) - a - P_{\text{Fix } L}(x - a) = L^n(x - a) - P_{\text{Fix } L}(x - a)$. “(ii) \Leftrightarrow (iii)”: This follows from [16, Lemma 2.6]. ■

Corollary 6.5. *Suppose that X is finite-dimensional. Let $L: X \rightarrow X$ be linear, nonexpansive and asymptotically regular, let $b \in X$, set $T: X \rightarrow X: x \mapsto Lx + b$ and suppose that $v_T := P_{\overline{\text{ran}(\text{Id} - T)}} 0$. Let $x \in X$. Then $v_T \in \text{ran}(\text{Id} - T)$ and*

$$T^n x + nv_T = (v_T + T)^n x = (T_{-v_T})^n x \rightarrow P_{\text{Fix}(v_T + T)} x \quad \text{linearly.} \quad (48)$$

Proof. Since X is finite-dimensional we learn that $\text{ran}(\text{Id} - T)$ is a closed affine subspace of X , hence $v_T \in \text{ran}(\text{Id} - T)$. Now Proposition 6.3 implies that $L^n x \rightarrow P_{\text{Fix } L} x$, which when combined with [16, Corollary 2.8] yields $L^n x \rightarrow P_{\text{Fix } L} x$ linearly. Now apply Proposition 6.4 ■

Theorem 6.6 (application to the forward-backward algorithm). *Suppose that A and B are affine and let $x \in X$. Then the following hold:*

- (i) $(T_{\text{FB}}(vA, B_v))^n x = (v + T_{\text{FB}})^n x = ((T_{\text{FB}})_{-v})^n x = T_{\text{FB}}^n x + nv$.
- (ii) If $v \in \text{ran}(\text{Id} - T_{\text{FB}})$ then

$$(v + T_{\text{FB}})^n x = ((T_{\text{FB}})_{-v})^n x = T_{\text{FB}}^n x + nv \rightarrow P_{\text{Fix}(v + T)} x = P_{Z_v} x. \quad (49)$$

- (iii) We have the implication

$$v = 0 \in \text{ran}(\text{Id} - T_{\text{FB}}) \Rightarrow T_{\text{FB}}^n x \rightarrow P_{\text{Fix } T} x = P_Z x. \quad (50)$$

If, in addition, X is finite-dimensional, then we also have

(iv) $v \in \text{ran}(\text{Id} - T_{\text{FB}})$ and

$$(v + T_{\text{FB}})^n x = ((T_{\text{FB}})_{-v})^n x = T_{\text{FB}}^n x + nv \rightarrow P_{\text{Fix}(v+T)} x = P_{Z_v} x \text{ linearly.} \quad (51)$$

(v) We have the implication

$$v = 0 \Rightarrow T_{\text{FB}}^n x \rightarrow P_{\text{Fix } T} x = P_Z x \text{ linearly.} \quad (52)$$

Proof. Proposition 3.4(ii) implies that T_{FB} is asymptotically regular and, since J_B is affine, (see [15, Theorem 2.1(xix)]) so is $v + T_{\text{FB}}$. (i): The first identity follows from (27) applied with w replaced by v . Now combine with Fact 6.1. (ii): Combine Proposition 6.3 and Corollary 4.7. (iii): This is a direct consequence of (ii). (iv) & (v): Combine Corollary 6.5 with (ii) and (iii), respectively. ■

Example 6.7. Let $L : X \rightarrow X$ be linear and firmly nonexpansive, let $b \in X$ and suppose that U is an affine subspace of X . Suppose that $A : X \rightarrow X : x \mapsto Lx + b$ and that $B = N_U$. Then the following hold¹⁸:

$$(i) \ Z_v = (v + U) \cap (L^{-1}((\text{par } U)^\perp - b + v)).$$

If, in addition, X is finite-dimensional then we also have:

$$(ii) \ \text{ran}(\text{Id} - T_{\text{FB}}) = \text{ran } L + (\text{par } U)^\perp + b.$$

$$(iii) \ v = P_{\text{par } U \cap \ker L} b.$$

Proof. (i): Let $x \in X$. Then $x \in Z_v \Leftrightarrow 0 \in Lx + b - v + N_U(x - v) = Lx + b - v + (\text{par } U)^\perp \Leftrightarrow [x - v \in U \text{ and } Lx \in (\text{par } U)^\perp - b + v] \Leftrightarrow [x \in v + U \text{ and } Lx \in (\text{par } U)^\perp - b + v] \Leftrightarrow x \in (v + U) \cap (L^{-1}((\text{par } U)^\perp - b + v))$. (ii): Using Theorem 5.3(ii) we have

$$\text{ran}(\text{Id} - T_{\text{FB}}) = \overline{\text{ran}}(\text{Id} - T_{\text{FB}}) = \text{ran } A + \text{ran } B \quad (53a)$$

$$= \text{ran } L + b + \text{ran } N_U = \text{ran } L + (\text{par } U)^\perp + b. \quad (53b)$$

(iii): Using Lemma 3.3(i) we learn that L is (maximally) monotone. Combining (ii), (11), (53), [27, Theorem 2.19] and [8, Proposition 20.17] we have

$$v = P_{\overline{\text{ran}}(\text{Id} - T_{\text{FB}})} 0 = P_{\text{ran } L + (\text{par } U)^\perp + b} 0 = b - P_{\text{ran } L + (\text{par } U)^\perp} b \quad (54a)$$

$$= P_{(\text{ran } L + (\text{par } U)^\perp)^\perp} b = P_{(\text{ran } L)^\perp \cap (\text{par } U)} b = P_{\ker L^* \cap \text{par } U} b = P_{\ker L \cap \text{par } U} b. \quad (54b)$$

■

Example 6.8 (MAP in the affine-affine feasibility case). Suppose that U and V are closed linear subspaces of X . Let $w \in X$. Suppose that $f = \frac{1}{2}d_{w+U}^2$, that $g = \iota_{w+V}$, that $A = \nabla f$ and that $B = \partial g$. Then $(\forall n \in \mathbb{N})$

$$(T_{\text{FB}})^n = (P_{w+V} P_{w+U})^n = (P_V P_U)^n (\cdot - w) + w. \quad (55)$$

Proof. Indeed, let $x \in X$. It follows from Example 3.9 applied with (U, V) replaced by $(w + U, w + V)$ and (11) that $T_{\text{FB}} = P_{w+U} P_{w+V} x = P_{w+V} (P_U(x - w) + w) = P_V (P_U(x - w) + w - w) + w = P_V P_U(x - w) + w$. Now (55) follows by simple induction. ■

¹⁸Suppose that U is a closed affine subspace of X . We use $\text{par } U$ to denote the parallel space of U defined by $\text{par } U := U - U$.

We now provide an application of the forward-backward algorithm that employs Pierra's product space technique introduced in [39]. For a general and more flexible framework of using the forward-backward algorithm to find a zero of the sum of more than two operators we refer the reader to the work by Combettes in [2, Section 2] and [26, Section 5].

Proposition 6.9 (application to parallel splitting). *Suppose that $m \in \{2, 3, \dots\}$. For every $i \in \{1, 2, \dots, m\}$, let $\alpha_i > 0$ and suppose that $A_i : X \rightarrow X$ are α_i -cocoercive. Set $\Delta := \{(x, \dots, x) \in X^m \mid x \in X\}$, set $\alpha = \min \{\alpha_i \mid i \in \{1, 2, \dots, m\}\}$, set $\mathbf{A} = \times_{i=1}^m \alpha A_i$, set $\mathbf{B} = N_\Delta$, set $\mathbf{T} = T_{\text{FB}(\mathbf{A}, \mathbf{B})}$, let $j : X \rightarrow X^m : x \mapsto (x, x, \dots, x)$, and let $e : X^m \rightarrow X : (x_1, x_2, \dots, x_m) \mapsto \frac{1}{m} (\sum_{i=1}^m x_i)$. Let $\mathbf{x} \in X^m$ and suppose that $\mathbf{v} := P_{\text{ran}(\text{Id} - \mathbf{T})} 0 \in \text{ran}(\text{Id} - \mathbf{T})$. Then the following hold:*

- (i) $\Delta^\perp = \{(u_1, \dots, u_m) \in X^m \mid \sum_{i=1}^m u_i = 0\}$.
- (ii) $\mathbf{Z}_\mathbf{v} := Z_{(\mathbf{v}, \mathbf{A}, \mathbf{B}_\mathbf{v})} = (\mathbf{v} + \Delta) \cap (\mathbf{A}^{-1}(\mathbf{v} + \Delta^\perp))$.
- (iii) $\mathbf{v} = 0 \Leftrightarrow \text{zer}(\sum_{i=1}^m A_i) \neq \emptyset$.
- (iv) X is finite-dimensional $\Rightarrow \text{ran}(\text{Id} - \mathbf{T}) \simeq \Delta^\perp + \times_{i=1}^m \text{ran } A_i$.

If $(\forall i \in \{1, 2, \dots, m\}) A_i$ is affine, then we additionally have:

- (v) $(\mathbf{v} + \mathbf{T})^n \mathbf{x} = (\mathbf{T}_{-\mathbf{v}})^n \mathbf{x} = \mathbf{T}^n \mathbf{x} + n\mathbf{v} \rightarrow P_{\text{Fix}(\mathbf{v} + \mathbf{T})} \mathbf{x} = P_{\mathbf{Z}_\mathbf{v}} \mathbf{x}$.
- (vi) X is finite-dimensional $\Rightarrow (\mathbf{v} + \mathbf{T})^n \mathbf{x} \rightarrow P_{\text{Fix } \mathbf{T}} \mathbf{x} = P_{\mathbf{Z}_\mathbf{v}}$ linearly.
- (vii) X is finite-dimensional $\Rightarrow \text{ran}(\text{Id} - \mathbf{T}) = \Delta^\perp + \times_{i=1}^m \text{ran } A_i$.

Proof. Note that $(\forall i \in \{1, \dots, m\}) A_i$ is α -cocoercive hence \mathbf{A} is firmly nonexpansive. (i): This is [8, Proposition 25.5(i)]. (ii): Let $\mathbf{z} \in X^m$. Then $\mathbf{z} \in \mathbf{Z}_\mathbf{v} \Leftrightarrow \mathbf{v} \in N_\Delta(\mathbf{z} - \mathbf{v}) + \mathbf{A}\mathbf{z} \Leftrightarrow [\mathbf{z} - \mathbf{v} \in \Delta \text{ and } \mathbf{A}\mathbf{z} - \mathbf{v} \in \Delta^\perp] \Leftrightarrow [\mathbf{z} \in \mathbf{v} + \Delta \text{ and } \mathbf{z} \in \mathbf{A}^{-1}(\mathbf{v} + \Delta^\perp)] \Leftrightarrow \mathbf{z} \in (\mathbf{v} + \Delta) \cap (\mathbf{A}^{-1}(\mathbf{v} + \Delta^\perp))$. (iii): It follows from (32), Proposition 3.4(iv) applied to \mathbf{A} and \mathbf{B} and (i) that $\mathbf{v} = 0 \Leftrightarrow \text{Fix } \mathbf{T} \neq \emptyset \Leftrightarrow (\exists \mathbf{z} \in X^m)$ such that $0 \in \mathbf{A}\mathbf{z} + N_\Delta \mathbf{z} = \mathbf{A}\mathbf{z} + \Delta^\perp \Leftrightarrow [\mathbf{z} \in \Delta \text{ and } \mathbf{A}\mathbf{z} \in \Delta^\perp] \Leftrightarrow [(\exists z \in X) \mathbf{z} = (z, z, \dots, z) \text{ and } \sum_{i=1}^m A_i z = 0] \Leftrightarrow z \in \text{zer}(\sum_{i=1}^m A_i)$. (iv): Apply Theorem 5.3(i) to \mathbf{A} and \mathbf{B} and note that $\text{ran } \mathbf{A} = \times_{i=1}^m \text{ran } A_i$. (v) & (vi): Apply Theorem 6.6(ii) and (iv) respectively to \mathbf{A} and \mathbf{B} . (vii): Apply Theorem 5.3(ii) to \mathbf{A} and \mathbf{B} . ■

7 Some algorithmic consequences

In this section we make use of the following useful fact that is well-known in analysis.

Fact 7.1. *Suppose that $(a_n)_{n \in \mathbb{N}}$ is a decreasing sequence of nonnegative real numbers such that $\sum_{n=0}^\infty a_n < +\infty$. Then*

$$na_n \rightarrow 0. \quad (56)$$

Proof. See [32, Section 3.3, Theorem 1]. ■

Lemma 7.2. *Let $L : X \rightarrow X$ be linear, nonexpansive and asymptotically regular, let $b \in X$, and suppose that $T : X \rightarrow X : x \mapsto Lx + b$ and that $v_T := P_{\text{ran}(\text{Id} - T)} 0 \in \text{ran}(\text{Id} - T)$. Let $x \in X$. Then the sequence $(\|T^n x - T^{n+1} x - v_T\|)_{n \in \mathbb{N}}$ is a decreasing sequence of nonnegative real numbers that converges to 0.*

Proof. Let $n \in \mathbb{N}$. It follows from Fact 6.1 that $T^n x + nv_T = (v_T + T)^n x$. Moreover, since L is nonexpansive so is $v_T + T$. Now

$$\begin{aligned}
\|T^n x - T^{n+1} x - v\| &= \|T^n x + nv_T - (T^{n+1} x + (n+1)v_T)\| \\
&= \|(v_T + T)^n x - (v_T + T)^{n+1} x\| \\
&\leq \|(v_T + T)^{n-1} x - (v_T + T)^n x\| \\
&= \|T^{n-1} x + (n-1)v_T - (T^n x + nv_T)\| \\
&= \|T^{n-1} x - T^n x - v_T\|.
\end{aligned} \tag{57}$$

The claim about convergence follows from Proposition 6.3. \blacksquare

Theorem 7.3. *Let $L: X \rightarrow X$ be linear, nonexpansive and asymptotically regular, let $b \in X$, and suppose that $T: X \rightarrow X: x \mapsto Lx + b$ and that $v_T := P_{\overline{\text{ran}(\text{Id} - T)}} \in \text{ran}(\text{Id} - T)$. Let $x \in X$ and set*

$$(\forall n \in \mathbb{N}) \quad x_n := T^n x + n(T^{n^2} x - T^{n^2+1} x). \tag{58}$$

Then $x_n \rightarrow P_{\text{Fix}(v_T + T)} x$.

Proof. We have

$$\begin{aligned}
\|x_n - (v_T + T)^n x\| &= \|T^n x + n(T^{n^2} x - T^{n^2+1} x) - (T^n x + nv_T)\| \\
&= n\|T^{n^2} x - T^{n^2+1} x - v_T\| = \sqrt{n^2}\|T^{n^2} x - T^{n^2+1} x - v_T\| \rightarrow 0,
\end{aligned} \tag{59}$$

where the limit follows by applying Fact 7.1 with a_n replaced by $\|T^n x - T^{n+1} x - v_T\|^2$. It follows from Proposition 6.3 that $(v_T + T)^n x \rightarrow P_{\text{Fix}(v_T + T)} x$, hence the conclusion follows. \blacksquare

Corollary 7.4. *Suppose that A and B are affine and that $v \in \text{ran}(\text{Id} - T_{\text{FB}})$. Let $x \in X$ and set*

$$(\forall n \in \mathbb{N}) \quad x_n := T_{\text{FB}}^n x + n(T_{\text{FB}}^{n^2} x - T_{\text{FB}}^{n^2+1} x). \tag{60}$$

Then $x_n \rightarrow P_{\text{Fix}(v + T_{\text{FB}})} x = P_{Z_v} x$.

Proof. Combine Proposition 3.4(i), Fact 2.3, Theorem 7.3 and Theorem 6.6(ii). \blacksquare

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References

- [1] H. Attouch and M. Théra, A general duality principle for the sum of two operators, *Journal of Convex Analysis* 3 (1996), 1–24.

- [2] H. Attouch, L. M. Briceño-Arias and P. L. Combettes, A parallel splitting method for coupled monotone inclusions, *SIAM Journal on Control and Optimization* vol. 48 (2010), 3246–3270.
- [3] J.B. Baillon, R.E. Bruck and S. Reich, On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces, *Houston Journal of Mathematics* 4 (1978), 1–9.
- [4] J.B. Baillon, Quelques propriétés de convergence asymptotique pour les contractions impaires, *Comptes rendus de l'Académie des Sciences* 238(1976), Aii, A587-A590.
- [5] J.-B. Baillon and G. Haddad, Quelques propriétés des opérateurs angle-bornés et n -cycliquement monotones, *Israel Journal of Mathematics* 26 (1977), 137–150.
- [6] H.H. Bauschke and W.M. Moursi, The Douglas–Rachford algorithm for two (not necessarily intersecting) affine subspace, *SIAM Journal in Optimization* 26, 968–985, 2016.
- [7] H.H. Bauschke, R.I. Boţ, W.L. Hare and W.M. Moursi, Attouch–Théra duality revisited: paramonotonicity and operator splitting, *Journal of Approximation Theory* 164 (2012), 1065–1084.
- [8] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, 2011.
- [9] H.H. Bauschke, F. Deutsch, H. Hundal and S.-H. Park: Accelerating the convergence of the method of alternating projections, *Transactions of the American Mathematical Society* 355 (2003), 3433–3461.
- [10] H.H. Bauschke, M.N. Dao and W.M. Moursi, On Fejér monotone sequences and nonexpansive mappings, *Linear and Nonlinear Analysis*, vol. 1, pp. 287–295, 2015.
- [11] H.H. Bauschke, W.L. Hare and W.M. Moursi, On the range of the Douglas–Rachford operator, *Mathematics of Operations Research*, in press.
- [12] H.H. Bauschke, W.L. Hare and W.M. Moursi, Generalized solutions for the sum of two maximally monotone operators, *SIAM Journal on Control and Optimization* 52 (2014), 1034–1047.
- [13] H.H. Bauschke and J.M. Borwein, Dykstra’s alternating projection algorithm for two sets, *Journal of Approximation Theory* 79 (1994), 418–443.
- [14] H.H. Bauschke, X. Wang and L. Yao, Rectangularity and paramonotonicity of maximally monotone operators, *Optimization* 63 (2014), 487–504.
- [15] H.H. Bauschke, S.M. Moffat and X. Wang, Firmly nonexpansive mappings and maximally monotone operators: correspondence and duality, *Set-Valued and Variational Analysis* 20 (2012), 131–153.
- [16] H.H. Bauschke, B. Lukens and W.M. Moursi, Affine nonexpansive operators, Attouch–Théra duality and the Douglas–Rachford algorithm, [arXiv:1603.09418 \[math.OA\]](https://arxiv.org/abs/1603.09418).
- [17] J.M. Borwein and J.D. Vanderwerff, *Convex Functions*, Cambridge University Press, 2010.
- [18] H. Brezis and A. Haraux, Image d’une Somme d’opérateurs Monotones et Applications, *Israel Journal of Mathematics* 23 (1976), 165–186.
- [19] H. Brezis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland/Elsevier, 1973.
- [20] R.E. Bruck and S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces, *Houston Journal of Mathematics* 3 (1977), 459–470.
- [21] R.S. Burachik and A.N. Iusem, *Set-Valued Mappings and Enlargements of Monotone Operators*, Springer-Verlag, 2008.
- [22] R.S. Burachik and V. Jeyakumar, *Journal of Convex Analysis* 12, (2005), 279–290.
- [23] P.L. Combettes, Solving monotone inclusions via compositions of nonexpansive averaged operators, *Optimization* 53 (2004), 475–504.
- [24] P. L. Combettes and V. R. Wajs, Signal recovery by proximal forward-backward splitting, *Multiscale Modeling and Simulation* 4 (2005), 1168–1200.
- [25] P. L. Combettes, Đinh Dũng and B. C. Vũ, Dualization of signal recovery problems, *Set-Valued and Variational Analysis* 18 (2010), 373–404.
- [26] P. L. Combettes and B. C. Vũ, Variable metric forward-backward splitting with applications to monotone inclusions in duality, *Optimization* 63 (2014), 1289–1318.
- [27] J. B. Conway, *A Course in Functional Analysis*, Springer-Verlag, 1990.
- [28] J. Eckstein, *Splitting Methods for Monotone Operators with Applications to Parallel Optimization*, Ph.D. thesis, MIT, 1989.

- [29] J.-P. Gossez, Opérateurs monotones non linéaires dans les espaces de Banach non réflexifs, *Journal of Mathematical Analysis and Applications*, 34 (1971), 371–395.
- [30] A.N. Iusem, On some properties of paramonotone operators, *Journal of Convex Analysis* 5 (1998), 269–278.
- [31] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, 1984.
- [32] K. Knopp, *Infinite sequences and series*, Dover, New York, 1956.
- [33] B. Lemaire, Which fixed point does the iteration method select? *Lecture Notes in Economics and Mathematical Systems* 452 (1979), 154–167.
- [34] P.L. Lions and B. Mercier, Splitting algorithms for the sum of two nonlinear operators. *SIAM Journal on Numerical Analysis* 16(6) (1979), 964–979.
- [35] G.J. Minty, Monotone (nonlinear) operators in Hilbert space, *Duke Mathematical Journal* 29 (1962), 341–346.
- [36] S.M. Moffat, W.M. Moursi and X. Wang, Nearly convex sets: fine properties and domains or ranges of subdifferentials of convex functions, *Mathematical Programming, Series A*, DOI: 10.1007/s10107-016-0980-z.
- [37] J.-J. Moreau, Proximité et dualité dans un espace hilbertien, *Bulletin de la Société Mathématique de France* 93 (1965), 273–299.
- [38] A. Pazy, Asymptotic behavior of contractions in Hilbert space, *Israel Journal of Mathematics* 9 (1971), 235–240.
- [39] G. Pierra, Decomposition through formalization in a product space, *Mathematical Programming* 28 (1984), 96–115.
- [40] R.T. Rockafellar and R.J-B. Wets, *Variational Analysis*, Springer-Verlag, corrected 3rd printing, 2009.
- [41] R.T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM Journal on Control and Optimization* 14 (1976), 877–898.
- [42] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.
- [43] R.T. Rockafellar, On the maximal monotonicity of subdifferential mappings, *Pacific Journal of Mathematics* 33 (1970), 209–216.
- [44] S. Simons, *Minimax and Monotonicity*, Springer-Verlag, 1998.
- [45] S. Simons, *From Hahn-Banach to Monotonicity*, Springer-Verlag, 2008.
- [46] B.F. Svaiter, On weak convergence of the Douglas–Rachford method, *SIAM Journal on Control and Optimization* 49 (2011), 280–287.
- [47] P. Tseng, Applications of a splitting algorithm to decomposition in convex programming and variational inequalities, *SIAM Journal on Control and Optimization* 29 (1991), 119–138.
- [48] E.H. Zarantonello, Projections on convex sets in Hilbert space and spectral theory, in: E.H. Zarantonello (Ed.), *Contributions to Nonlinear Functional Analysis*, Academic Press, New York, (1971), 237–424.
- [49] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/A: Linear Monotone Operators*, Springer-Verlag, 1990.
- [50] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/B: Nonlinear Monotone Operators*, Springer-Verlag, 1990.
- [51] E. Zeidler, *Nonlinear Functional Analysis and Its Applications I: Fixed Point Theorems*, Springer-Verlag, 1993.